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MEASUREMENT OF FIELD STRUCTURE [illegible]
BY THE PERTURBATION [illegible] RADIATION

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Summary

A new indirect method for measuring the absolute values of the field strength on the boundaries of microwave resonators is presented. This method is based on the perturbation of the radiation field, which can be induced by changing the surface impedance of the resonator's conducting boundaries. The method is simple as well as accurate for applications where the other indirect methods, based on the perturbation of the mode-field, fail or become laborious and inaccurate.

The perturbation formulas used to calculate the tangential field are simple and were readily derived from the classical theory of propagation of electromagnetic waves in media of high (but finite) conductivity.

The new method was used to determine the distribution of surface current on the boundaries of a klystron re-entrant resonator. Relevant experimental results are presented.

I. Introduction

The analytical solution of the electromagnetic fields in microwave structures can be derived rigorously for only a very limited number of shapes that are definable in terms of one of the standard coordinate systems. For structures with relatively complicated shapes, approximate methods, such as the variational or equivalent static methods, are sometimes used. However, the accuracy of such approximate analytical methods is often unsatisfactory, and in many practical applications numerical or experimental methods have been found to give more accurate results.

On the other hand, while numerical methods such as the relaxation or netpoint method[1] have been greatly facilitated by the advent of high speed computers, these methods still require a considerable amount of computation because the values of the field everywhere inside the microwave structure have to be determined simultaneously (such detailed information is not required in many cases).

The various experimental methods that have been developed for measuring the fields in microwave structures can be divided into two main categories: the direct or sampling methods and the indirect or perturbation methods. By the direct methods, measurements can be made more simply and little or no computation is required to calculate the field strengths from the measured results. On the other hand, the sampling can be done only near the structure boundaries, and therefore, only relative values of the field strengths near the boundaries can be determined. It is sometimes possible to calibrate the sampling indicator so that it can measure absolute values of the fields; however, such a calibration usually deprives the sampling methods of their simplicity.

Furthermore, in order to obtain accurate results, great care and precision in the design and construction of the sampling mechanism is required, and such precision can be difficult to achieve in the case of nonplanar surfaces.

The indirect methods are based on the theorems which give the change in the resonant frequency of a cavity resonator when its fields are perturbed. The electromagnetic fields in a resonator can be divided into two types: the mode-field, which contains the reactive energy, and the radiation field, which represents the dissipated or resistive energy. All the existing indirect methods of measuring field strength are based on the principle that perturbation of the mode-field results in a shift in the real part of the resonant frequency. In the method developed by Hansen,[2] the field strengths are measured near the resonator boundaries by deforming a small part of the resonator wall in such a way that, at the limit of small deformation, the new surface is parallel to the original surface. In this case, the perturbation equation is given by

$$\frac{\Delta\lambda}{\lambda_o} = \frac{1}{2} \frac{\int_{\Delta v} (\epsilon E^2 - \mu H^2)\, dv}{\int_v \mu H^2\, dv}, \tag{1}$$

where λ_o is the unperturbed wave length, $\Delta\lambda$ is the shift in the resonant wavelength due to an incremental deformation of volume Δv, **E** and **H** are the peak fields of the unperturbed cavity, and μ and ϵ are the permeability and permitivity, respectively. In general, both **E** and **H** are unknown, and, therefore, this perturbation equation [Eq. (1)] will be useful only when either the electric or the magnetic field component is known to be negligibly small.

In another method, devised by Slater and Maier[3] for measuring the fields in the interior of a resonator, the electric and magnetic field components were measured separately by taking advantage of the fact that obstacles of different shapes suspended inside the resonator, react differently to the two field components. The perturbation formulas for metallic and dielectric ellipsoids of revolution in the three special cases, sphere, needle and disc, have been derived by various workers and are summarized in a report by Mullett[4] (which also includes a list of the important references). In deriving these formulas, it was assumed, first, that the perturbing ellipsoid produces a small change in the over-all geometrical configuration of the field inside the resonator, and, second, that the field in the neighborhood of the ellipsoid approximates the static field. Therefore, if the derived formulas are to be used in accurate measurements, the perturbing ellipsoid should be small enough for the unperturbed field to be regarded as uniform for an appreciable distance around the ellipsoid. (It was found experimentally[3] that in order to limit the error to 1 per cent, the size of the perturbing metallic sphere should be of the order of 10^{-5} cm^3 at S-band frequencies.)

It is possible to construct small metallic ellipsoids of spherical shape with good precision; however, the other two ellipsoidal shapes and the dielectric sphere are not as simple to construct, and the small size of the perturbing ellipsoid becomes the factor limiting the accuracy of the method at high microwave frequencies. Another difficulty in using the Slater method is encountered in measuring the field near the conducting boundaries of the resonator. This difficulty is due to the fact that the derived perturbation formulas do not apply when the object is close to the conducting walls because multiple reflections between the perturbing object and the walls become too strong to be neglected. It is sometimes possible to correct for such proximity effects by applying the theory of images; however, such a correction increases the complexity of computation and degrades the approximation of the perturbation formulas. The method described in this report overcomes some of the difficulties mentioned above and has the advantages of simplicity and accuracy.

II. Field Measurement by the Perturbation of the Radiation Field

The indirect methods discussed in the previous section are based on the perturbation of resonator fields by an object made of a perfect metal or a perfect dielectric. These types of perturbation result in a change of the reactive energy associated with the mode-fields, and this change can be detected by measuring the shift in the real part of the resonant frequency. One of the major justifications for developing these methods was the ease and accuracy with which small changes in the resonant frequency could be measured. Unfortunately, the accuracy of field strengths determined by these methods was found to be limited by the precision with which the perturbing object was constructed and not by the accuracy of the electrical measurements. This fact is not surprising since, in the search for simple solutions of the perturbation equations, stringent requirements had to be imposed on the shape and size of the perturbing object. It was also found[5] that the mode-field perturbation methods are laborious and inaccurate when used for measuring the fields near resonator boundaries. A satisfactory solution of measuring the absolute field strengths near the resonator walls can be achieved by using the radiation field perturbation method. For instance, if instead of perturbing the mode-field near the resonator boundary, we perturb the field which radiates through the boundary (by changing the wall surface impedance), the result will be a shift in the imaginary part of the resonant frequency, which can then be detected by measuring the change in the resonator unloaded Q.

The main advantage of the radiation-field perturbation method is that the restriction on the amount of perturbation in the radiation-field method is not as severe as it is in the mode-field perturbation method. For instance, in the case of mode-field perturbation methods, the shift in the resonant frequency must be limited to a fraction of one per cent if the error is to be kept within a few per cent; in the case of the radiation-field method, it is possible to perturb the field so as to change the Q by almost one order of magnitude without any appreciable error in applying the perturbation equation.

In order to obtain the perturbation equation for the radiation field on the resonator boundaries, we may start with the general expression for the resonator unloaded Q, which is given by Eq. (4-A) in the Appendix.

$$Q_u = \frac{2\mu_o \int_v H_v^2 \, dv}{\sum_{r=1}^{n} \mu_r \delta_r \int_{S_r} H_r^2 \, dS_r},$$

where

μ_o = permeability of medium filling the resonator,

μ_r = permeability of rth wall,

δ_r = skin depth of the rth wall,

H_v = magnetic field in the interior of the resonator,

H_r = surface magnetic field on the rth wall,

S_r = surface area of rth wall.

If we assume an average value $\overline{H_r^2}$ for the square of the surface magnetic field on the rth wall, then Eq. (4-A) for a nonferrous resonator ($\mu_r = \mu_o$) reduces to

$$Q_u = \frac{2 \int_v H_v^2 \, dv}{\sum_{r=1}^{n} \delta_r \, (\overline{H_r^2} \cdot S_r)} . \tag{1}$$

When all the walls have the same surface impedance, i.e., the same skin depth δ, Eq. (1) becomes

$$Q_u = \frac{2 \int_v H_v^2 \, dv}{\delta \sum_{r=1}^{n} \overline{H_r^2} \cdot S_r} . \tag{2}$$

Now, if we change the surface impedance of the nth wall by changing the skin depth from δ to δ_n and assume that such a change has a negligible effect on the mode-field configuration, then the Q of the new perturbed cavity will become

$$Q_u^n = \frac{2 \int_v H_v^2 \, dv}{\delta \left[\sum_{r=1}^{n-1} (\overline{H_r^2} \cdot S_r) \right] + \delta_n \overline{H_n^2} \cdot S_n} . \tag{3}$$

From Eqs. (2) and (3) we get

$$\frac{Q_u}{Q_u^n} = \frac{\delta\left[\sum_{r=1}^{n-1} \overline{(H_r^2} \cdot S_r)\right] + \delta_n \overline{H_n^2} \cdot S_n}{\delta \cdot \sum_{r=1}^{n} \overline{(H_r^2} \cdot S_r)}$$

$$= 1 + \frac{(\delta_n - \delta)\, \overline{H_n^2} \cdot S_n}{\delta \cdot \sum_{r=1}^{n} \overline{(H_r^2} \cdot S_r)} \qquad (4)$$

Equation (4) gives the average value of the square of the magnetic field on the surface of the nth wall in terms of the average magnetic field on all the walls of the resonator. By carrying out the same procedure for the other walls, one by one, the average value of the magnetic field everywhere on the resonator boundaries can be determined. (The absolute value of the field can be determined by measuring the input power to the resonator.)

In deriving Eq. (4), we made the assumption that the change in the spatial phase (geometrical configuration) of the mode-field was negligibly small. It is true that perturbation of the wall surface impedance can result in a very large change in the magnitude of the mode-field H_v for a given amount of power injected into the resonator. However, if the geometrical configuration of the mode-field does not change with the change of the surface impedance, then the magnitude of the radiation field H_s will change by exactly the same ratio as that of the mode-field, and Eq. (4) will be exact. Therefore, any error in applying Eq. (4) will be due only to the perturbation of the geometrical configuration of the mode-field. In practice, however, any change in the wall surface impedance would result in the perturbation of the mode-field geometrical configuration, and it would be necessary, therefore, to find the order of approximation in applying Eq. (4).

Two types of perturbation of the mode-field configuration can result from a change in the wall surface impedance. The first is due to the change in the stored (reactive) energy inside the conducting walls, while the second is due to a possible change in the path of the surface conduction currents. The first type is inherent in the system and is proportional to the change in the imaginary part of the surface impedance. However, as shown in the Appendix, the energy stored in the walls of a moderately high Q resonator is often very small and can be either neglected or computed and used as a correction term.

The second type of mode-field perturbation caused by the possible change in the path of the surface currents can be relatively large, and therefore should be avoided or minimized if the method described here is to be used. In order to avoid this type of mode-field perturbation it would be necessary to make sure that a change in the surface impedance did not produce new paths for the surface currents. For example, in the case of the E_{010} - mode in cylindrical resonators, the change in the surface impedance of an annular ring [Fig. 1(a)] should cause no change in the path of the surface currents. On the other hand, a change in the surface impedance of a longitudinal strip such as that shown in Fig. 1 (b) might cause some of the current to be forced out of the strip if its surface impedance were higher than that of the neighboring parts of the wall, or force the current into the strip if its surface impedance were lower. It would, therefore, seem necessary to have a knowledge of the general configuration of the surface current before perturbing the radiation field. However, since most engineering microwave structures have some sort of symmetry, it would not be difficult to change the surface impedance so as to perturb the radiation field without seriously perturbing the mode-field. It is also possible to record any large change in the mode-field configuration resulting from a change in the surface impedance by measuring the shift in the resonant frequency. Such a record, which can be made during the Q measurements, will provide a means of determining whether the perturbations are properly placed. Furthermore, if after perturbation the Q of the resonator is still high (of the order of 1000), and if the change in the surface impedance is small (one or two orders of magnitude), then the distribution of the surface current will be determined mainly by the mode-field and will be very little affected by the change of the wall surface impedances. Therefore, under the conditions of high Qs and small changes in the surface impedance, the error due to an incidental mode-field perturbation will in general be small in comparison with the experimental error. (This fact was demonstrated in the course of our experimental investigations, but a discussion of the pertinent experiments is beyond the scope of this report.)

III. Experimental Results

1. Q Measurements:-

The main purpose of the experimental work was to measure the field strength on the boundaries of a cylindrical re-entrant resonator of

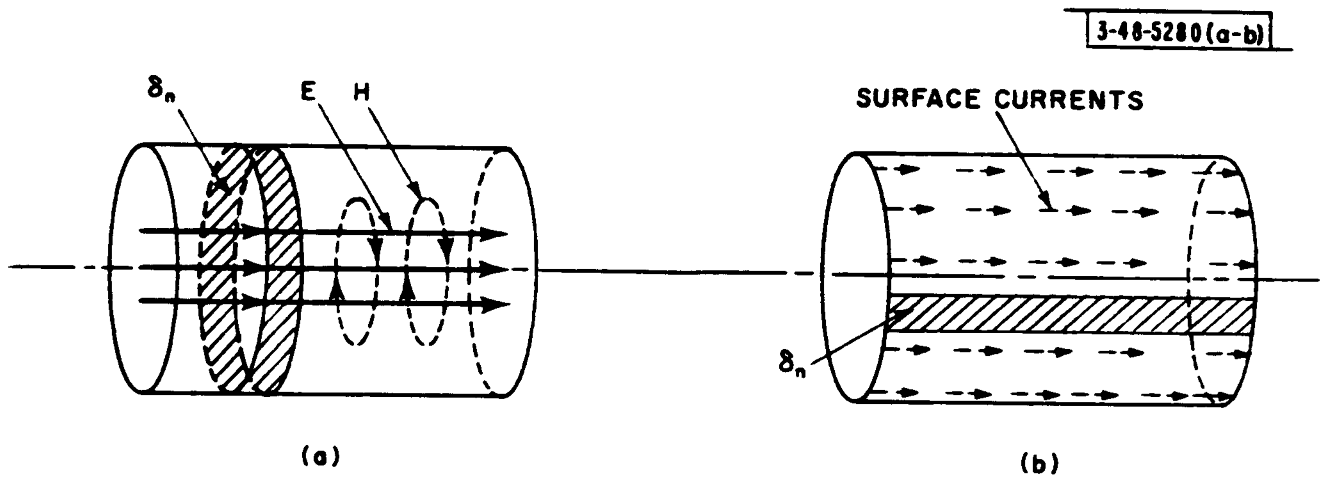

Fig. 1. Two different types of surface impedance perturbation in a cylindrical resonator supporting E010 mode.

(a) Symmetric perturbation does not perturb the mode-field.

(b) Asymmetric perturbation perturbs the mode-field.

the type used in high power klystrons at S-band frequencies. The result of such experiments would be useful for practical purposes For example, at high average powers the dissipation of the heat due to ohmic losses on the walls of the output resonator is often a problem and there is a need for the design of an efficient cooling system based on knowledge of the distribution of the RF surface currents. Also, it has been suggested that materials with low secondary emission be used at the gap edges of klystron resonators in order to reduce or eliminate multipactor. The RF resistivity of these materials is usually much higher than that of copper, the material normally used[8], and therefore a knowledge of the absolute value of the local field strength would be required in order to evaluate the extra losses due to the low secondary materials.

The accuracy of the radiation field perturbation method is determined primarily by the accuracy with which the resonator unloaded Q is measured. Our goal was to be able to measure the field strength with an accuracy of 1 per cent; that is, the values of Q, which are expected to range from approximately 1000 to 10,000, should be accurate within 1 per cent.

There are several methods[9] for measuring the Q, each of which is most suitable in a certain range of Q values and for a certain application. Two of these methods which seemed to be suitable for the present application were tried. The first is the conventional method of measuring the bandwidth on the response curve of frequency vs the power transmitted through the resonator. The second method, relatively new,[10] is based on the phase shift produced by the resonator at off-resonance frequencies; unlike the conventional method of measuring the phase shift at microwaves, this method transfers the phase measurements to low frequencies by employing variable frequency amplitude modulation. The advantages of the phase method are: first, the Q measurements are not sensitive to small shifts in the microwave carrier frequency and, second, the bandwidth being related directly to the low modulation frequency can be measured more accurately and easily than if it were the difference between two high microwave frequencies. Unfortunately, some of our Q measurements would have required phase measurements at frequencies as high as one or two megacycles, and because no suitable high frequency phase meter was available,* it was decided that the first method should be used throughout the course of measurements.

A setup for Q measurements using the amplitude -vs- frequency method is shown in the block diagram of Fig. 2. The stalo is a home-made passive stabilized S-band klystron oscillator with frequency stability of the order of one part in 10^6 over a period of a few minutes. The IF oscillator (a Hewlett-Packard instrument, model 606A) has frequency stability better than one part in 10^5, but since the IF frequency is only about one hundredth of the stalo frequency, the frequency stability of the whole system is determined by the stalo stability. The accuracy of the frequency measurements is determined by the frequency stability of the transfer oscillator (a Beckman/Berkeley instrument, model 7580) which is of the same order as that of the stalo.

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*The only commercial phase meter available was the AD-YU205A, which has a specified accuracy better than the required $1/4^{\circ}$, but needs continuous adjustments and tuning which make the measurements rather difficult and laborious.

For amplitude measurements, both the comparison and the direct methods were used. In the direct method, two 1-Kcps standing wave indicators (Hewlett-Packard 415B) were used to monitor the input and output powers of the resonator. In the comparison method, two instruments were used; the first was a DC differential voltmeter for CW operation, and the second was a ratio meter (Hewlett-Packard 416A) for 1-Kcps AM operation. All the different methods gave results for the Q measurements repeatable within 1 per cent.

The measurements procedure was to set the IF frequency f_i to about 30 Mcps and then adjust the stalo frequency f_o for the upper sideband frequency $(f_o + f_i)$ so that it was equal to the resonant frequency of the resonator. The IF frequency was chosen so that when the filter was tuned to the upper sideband frequency $(f_o + f_i)$, the lower sideband frequency $(f_o - f_i)$ was sufficiently rejected. The amplitude frequency response of the resonator was obtained by varying the IF frequency and measuring the transmission loss, using the precision calibrated attenuator to keep constant output power. The Q was then calculated from the half-power bandwidth and the resonant frequency. The measurements always resulted in a smooth response curve, and it was found by experience that the Q measurements can be very much simplified, without sacrificing accuracy, by making only two measurements at the half-power frequencies.

2. Resonator Construction:-

The cylindrical re-entrant resonator was constructed out of four machined walls which were screwed tightly together. In constructing a resonator in this way, care must be taken to ensure good electrical con-

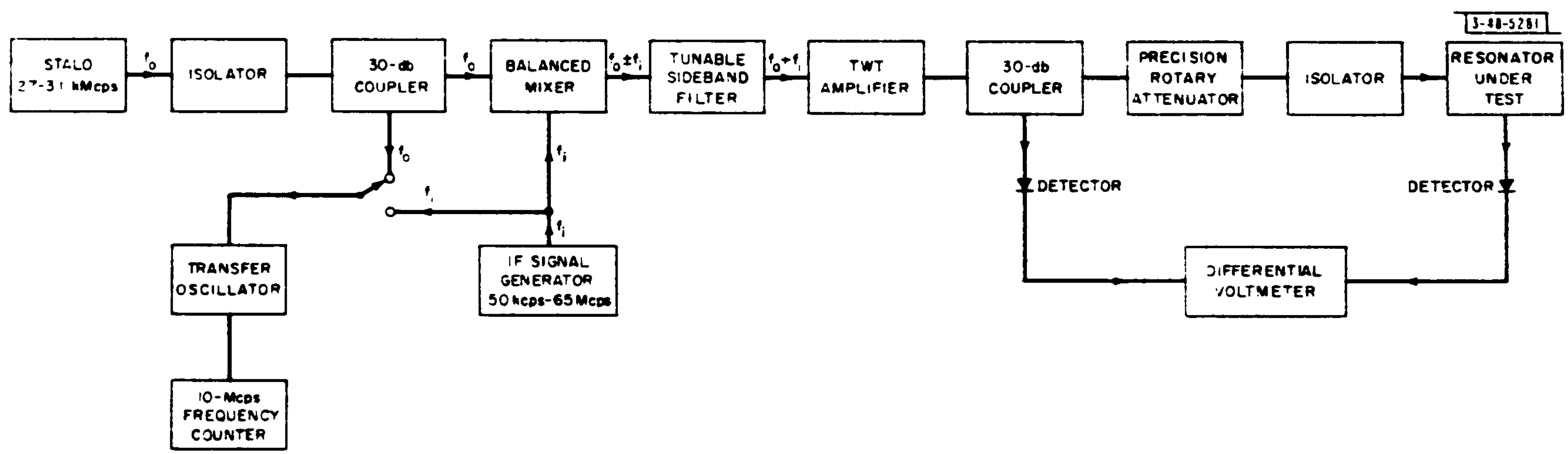

Fig. 2. **Q-Measurement by amplitude-vs-frequency method.**

tact between the assembled parts. This was achieved by avoiding large surface contact areas between the different walls, and by using as many screws as was practical. Figure 3 shows how large contact areas between the resonator walls were avoided, and Fig. 4 is a photograph of the unassembled cavity. No other special consideration was given to the mechanical design or to the machining processes. The resonator was reassembled several times, and the results of the Q measurements were reproducible within the error of the electrical measurements (less than one per cent).

The input to the resonator was by magnetic coupling through a 1/4" hole in the cylindrical wall (Wall No. 2), while the output was coupled to a crystal detector (Hewlett-Packard 444A) by an electric probe in the top cover (Wall No. 4). Both the input and output couplings were continuously adjustable by rotating the loop at the end of the input coaxial feeder and the withdrawal of the output probe-detector assembly. The loadings of both the input and output of the resonator were reduced until the change in the measured values of the Q was less than the experimental error.

When the surface impedance of the total area of any of the four walls of the copper resonator was to be changed, the wall was replaced by a similar wall made of different metal (such as brass or stainless steel), but when the surface impedance of only part of the wall was to be changed, the spraying technique was found to be most suitable. The spraying, done by plasma torch, results in a rather rough surface, but since the surface impedances of the sprayed materials were to be determined experimentally at the operating frequency, no attempt was made to improve the surface conditions, and the only precaution taken

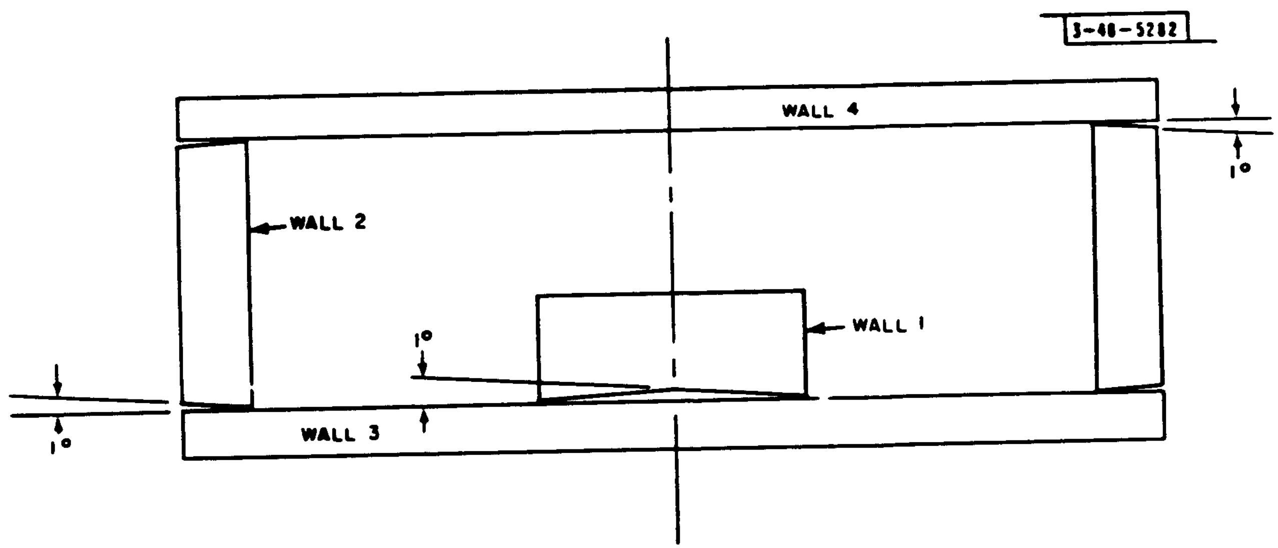

Fig. 3. Four-wall cylindrical reentrant resonator.

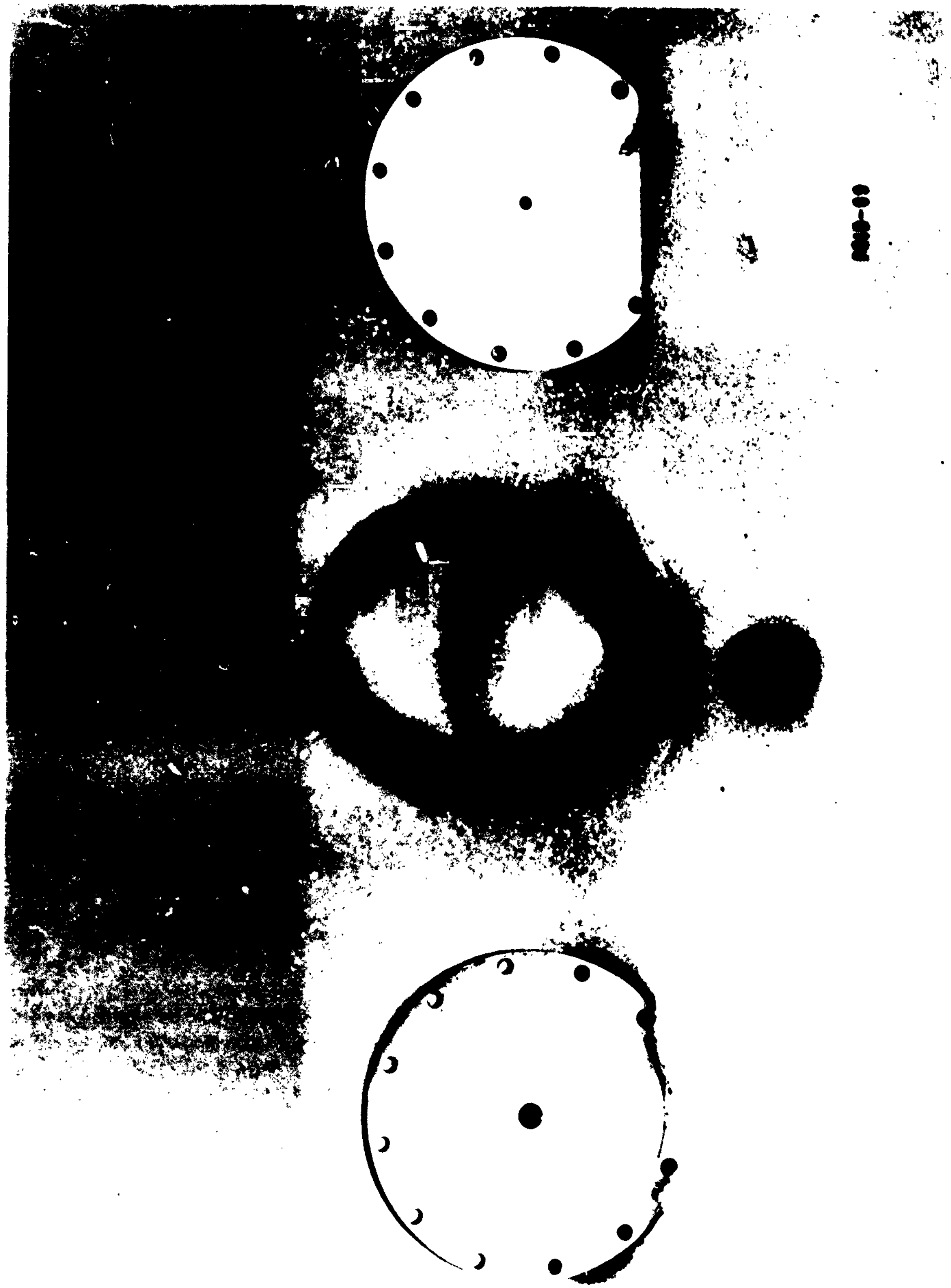

Fig. 4. Four walls of reentrant resonator under test.

was to ensure uniform coatings.

3. Measurements:-

First, measurements were made to determine the percentage of ohmic loss on each of the four walls of the resonator. For this purpose two similar resonators were constructed, one made of copper and the other of brass. The unloaded Qs of the two resonators were measured. The results were 7950 for the copper resonator and 4000 for the brass resonator. Thus, the effective skin depth for brass at the operating frequency (approximately 3000 Mcps) is 1.97 times the skin depth of copper. The copper and brass walls were then interchanged to form composite brass-copper resonators and their Qs were measured. In order to check the results, measurements were taken on two sets of four composite resonators. The first set consisted of a resonator made of three copper walls and one brass wall; the second set consisted of a resonator made of three brass walls and one copper wall. The loss on each of the four walls of each resonator was calculated by using Eq.(4) of Sec. II. The sum of the losses on the four walls was 101.5 per cent for the first set of cavities, and 99.5 per cent for the second set. The mean values of the two sets of measurements are shown in Fig. 5.

Next, the field strength on wall No. 1 (drift tube wall) was to be measured. However, some work done at that time on the stalo to improve its frequency stability also changed its frequency range. As a result, the resonator dimensions had to be changed so that the resonant frequency would fall within the new frequency range of the stalo. The change was accomplished by increasing the length of Wall No. 1 from 0.300" to 0.324". This increase changed the wall losses from 15.3 per cent to 19.4 per cent. The main purpose of the experiments was thus to determine how the 19.4 per cent of the total field is distributed on the drift tube wall. In these experiments the spraying technique was used to change the wall surface impedance. The idea was to spray the total wall surface area to a thickness of about 0.002". The coating was then machined off all the surface except for an annular ring (Fig. 6) where the field strength was to be determined. From the Q of the resonator with the sprayed ring wall and the Q of the all-copper resonator, the local field strength at the ring was determined. When a large number of measurements were required, as in plotting the field distribution over the wall surface, this method was found to be rather laborious, and another method was used. In the second method, the surface impedance at the place where the field had to be determined was changed from that of the sprayed material to that of copper, the base metal, which is the reverse of the first method.

The procedure for plotting the field distribution was as follows: starting using a wall with its surface completely sprayed, a narrow annular ring (about .030" wide) of the coating was machined off the completely sprayed wall, thus changing the surface impedance of the ring to that of copper. The Qs of the resonator were measured before and after the ring of coating was machined off the wall. The difference between the two values is a measure of the local field strength at the ring. The width of the machined ring was then gradually increased and the Q measured each time, until all the coating was machined off. In order to check the results, two sets of measurements were taken on two identical walls sprayed with TiC. In the first set of measurements, we started the cuts at the gap edge, and in the second set the cuts were

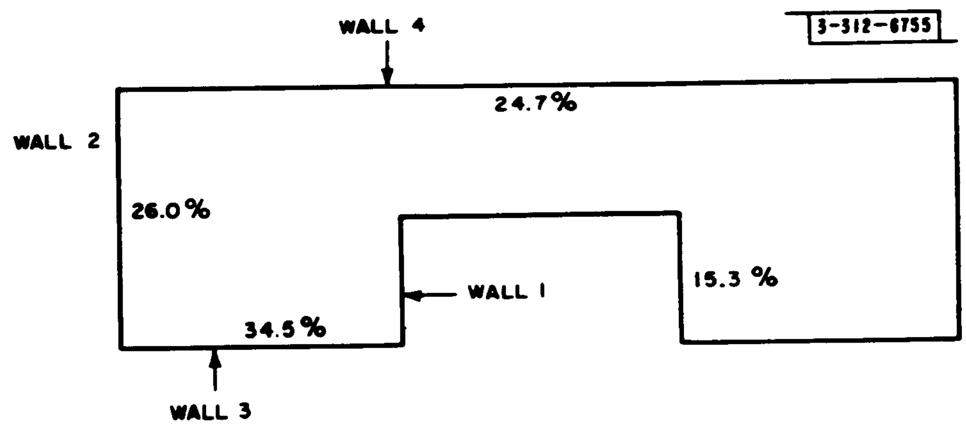

Fig. 5. Average ohmic losses on the four walls of cylindrical reentrant resonator.

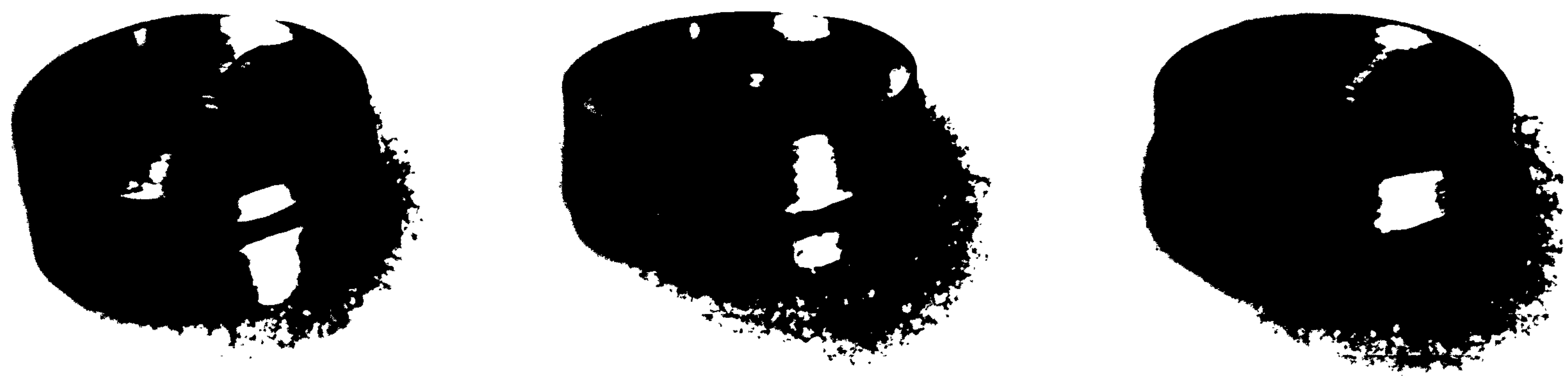

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Fig. 6. Changing the surface impedance of resonator walls at different positions.

started on the other end of the wall. The results of these experiments are shown in Fig. 7.

V. Conclusion

A new indirect method for measuring the absolute values of the strength of electromagnetic fields at microwave frequencies has been presented and its advantages and limitations have been discussed. By comparison with the other methods being used, the new method was found to be simpler and more accurate in certain applications. It was also found to be more flexible. For example, with the new method, it is possible not only to obtain detailed information about the field strengths, but to measure directly, with one single measurement, the average value of the field over a specific part of the surface.

It was not possible to compare the experimental results with theoretical results, since the solution of the field inside the resonator is not known. A method for solving the field inside the resonator numerically is being developed, and a comparison of the theoretical and experimental results will be presented in another report. (It would have been possible to make the field measurements on a resonator in which the fields are known theoretically, however we were interested in obtaining data on the fields in a typical klystron resonator.)

The method as presented is suitable primarily for measuring the field on resonator boundaries. However, the method may be modified so as to measure the two field components in the interior of resonators. For instance, by using a lossy perturbing sphere two quantities can be measured: the shift in the resonant frequency due to the perturbation

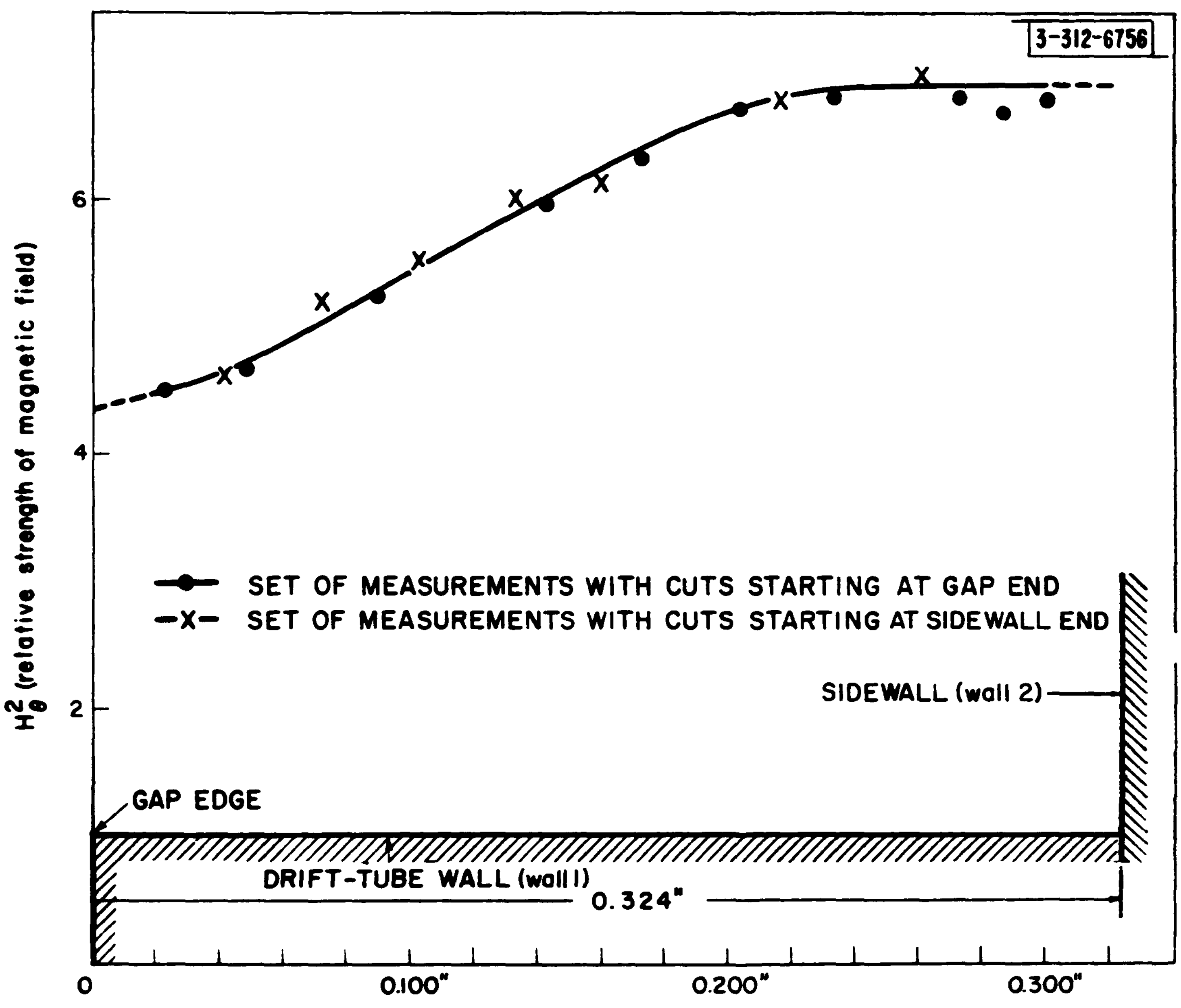

Fig. 7. Surface field distribution on the drift-tube wall of reentrant resonator.

of the mode-field, and the change in Q due to radiation-field perturbation. The perturbation equations for the lossy sphere have not been derived, but they should not be very difficult to derive if the need arises.

Appendix

The Q of a Composite Wall Resonator

The Q of a resonator is commonly defined as:

$$Q = 2\pi \frac{\text{Energy stored at resonance}}{\text{Energy lost per cycle}}$$

$$= \frac{\omega_o U}{W_L}, \qquad (1\text{-A})$$

where ω_o = angular frequency at resonance,

U = total energy stored in the resonator,

W_L = average power loss,

The stored energy inside the resonator -- sometimes referred to as the reactive energy by analogy with low frequency - is associated with the resonator mode-fields. In a loss-free resonator, the total energy stored in a certain mode would be oscillating between the electric field and the magnetic field at the mode resonant frequency. Therefore, the total energy stored in a hypothetical lossless resonator can be calculated from either the electric or magnetic field at their maxima:

$$U = \int_v \frac{1}{2} \mu H^2 dv = \int_v \frac{1}{2} \epsilon E^2 \, dv, \qquad (2\text{-A})$$

where E and H are the peak values of the electric and magnetic fields, respectively; μ, ϵ are the permeability and permitivity of the medium filling the resonator; and the integration is carried over the total resonator volume.

In a practical resonator, there are always some ohmic losses in both the dielectric and conductor media. For an air-filled resonator, the losses in the air medium, at any of the resonant modes, are negligibly small compared with the losses in the best conductor known at present. In such a resonator, part of the magnetic energy would be transferred to ohmic losses, and therefore, Eq. (2-A) cannot be exact for a practical resonator.

In order to find the exact amount of stored energy as well as the conductor losses, it would be necessary to solve the wave equations in both the air (as an ideal dielectric), and the conductor media, and match the two solutions on the boundaries (tangential fields must be continuous). The solution would be very difficult for almost all practical resonators, and in particular for the present application where the solution of the wave equation is not known even for the lossless resonator. However, if the losses are small in comparison with the stored energy, it is possible to apply the small-perturbation theory by assuming that the mode-fields in the practical resonator are the same as for the lossless resonator; that is, Eq.(2-A) is valid as a first approximation. (The error involved in the use of this first approximation will be discussed later.)

Assuming a certain distribution of surface magnetic field the losses due to the finite conductivity of the resonator walls are well known and are given in several text books [7, 9]. For a resonator with n walls of areas S_1, S_2, S_r --- S_n, skin depths δ_1, δ_2 -- δ_r -- δ_n, and permeabilities μ_1, μ_2 --μ_r -- μ_n would be given by:

$$W_L = \sum_{r=1}^{n} \int_{S_r} \frac{1}{2} H_r^2 \frac{\omega\mu\delta}{2} dS_r \qquad (3\text{-A})$$

Substituting Eqs. (2-A) and (3-A) into Eq. (1-A), we obtain the unloaded Q for a composite wall resonator:

(3-A)

$$Q_u = \frac{\omega \int_v \frac{1}{2}\, \mu_o \; H^2 \, dv}{\sum_{r=1}^{n} \int_{S_r} \frac{1}{2}\, H_r^2 \, \frac{\omega\mu\delta}{2} \, dS_r} ,$$

$$= \frac{2\mu_o \int_v H^2 \, dv}{\sum_{r=1}^{n} \mu_r \delta_r \int_{S_r} H_r^2 \, dS_r} \qquad \text{(4-A)}$$

In deriving Eq. (4-A) we have assumed, as a first approximation, that the energy stored is independent of the surface impedance of the resonator and is equal to the mode-field energy of a hypothetical lossless resonator. As a second approximation, we may consider the stored energy equal to the mode-field energy plus the small amount of energy which is stored in the field radiating into the conducting walls and is being transferred to ohmic energy.

To calculate the energy stored in the radiation field, we can either use Poynting's theorem and take the imaginary part of the Poynting vector as the energy stored, or we may find an equivalent lumped surface impedance from which we calculate the reactive energy. Taking the second approach, we have for the current density at a distance x from the conducting surface

$$i_x = i_s e^{-\gamma x} ,$$

where i_s is the current density on the surface, and

$$\gamma = (1 + j)/\delta .$$

The total current in the conducting wall (assumed to be of infinite thickness) is then

$$I = \int_0^\infty i_s e^{-\frac{1+j}{\delta} x} = i_s \frac{\delta}{1+j} \quad . \tag{5-A}$$

But from Ohm's law the electric field on the conductor surface is related to the current density at the surface by the relation

$$E = \frac{i_s}{\sigma} \quad . \tag{6-A}$$

From Eqs. (5-A) and (6-A) we get for the surface impedance per square

$$Z_s = \frac{E}{I} = \frac{i_s/\sigma}{i_s \delta/(1+j)} = \frac{(1+j)}{\delta\sigma} \quad ,$$

$$\therefore\ R_s = X_s = \omega L = 1/\delta\sigma \quad ,$$

where L is an equivalent lumped inductance.

Therefore, the average power loss is given by

$$W_L = \frac{1}{2} R_s I^2 = \frac{1}{2} \omega L I^2 \quad ,$$

$$= \omega \cdot \left[\frac{1}{2} L I^2\right] \quad ,$$

but $1/2\ LI^2$ is the magnetic energy stored in the equivalent lumped inductance which we will call ΔU.

Therefore, the total stored energy in the resonator will be given by

$$U + \Delta U = U + W_L/\omega \quad ,$$

and the unloaded Q for the second approximation becomes

$$Q_u' = \frac{\omega \cdot U + W_L}{W_L} = Q_u + 1 \quad . \tag{7-A}$$

For a high Q resonator, the difference between the two expressions of Q as given by Eq. (1-A) for a first approximation and Eq. (7-A) for a second approximation is very small and in most cases is less than the experimental error.*

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*The result of this second approximation would agree with the correct result obtained from transmission line theory. For a loss free dielectric transmi ssion line the Q is given by: $Q = \beta/2\alpha = \omega L/R\ [1 + 1/4(R/\omega L)^2 + \text{higher order terms}]$ where α and β are respectively the real and imaginary parts of the propagation constant and L and R are the equivalent distributed series inductance and resistance. Although the second term in the expansion is of the order of $1/Q^2$ ($Q \simeq \omega L/R$), the equivalent inductance L of the lossy line differs from that of the lossless line and the contribution due to this difference can be shown to be of the order of $1/Q$.

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